Supplement to ‘Extension of the Dip-test Repertoire - Efficient and Differentiable p-value Calculation for Clustering’

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1 TailoredDip

In the following we explain TailoredDip, which adds two extensions to the UniDip [7] algorithm. First, we show how a cluster can be expanded to include the tails of a distribution and then how outliers can be assigned to an appropriate cluster.

1.1 Capturing the Tails As mentioned in the paper, UniDip has problems correctly identifying the tails of distributions. These are usually labeled as noise. This behavior can be observed in Fig. 1a. The Gaussian clusters are not completely captured, but only the densest parts of the distributions. The same applies when uniformly distributed noise is added to the data (see Fig. 1c). TailoredDip is superior in capturing the tails of the distributions in both cases. This is also confirmed by the normalised mutual information (NMI) score and can be seen in Fig. 1b and 1d. We achieve this improvement by checking the spaces between the clusters for additional structures after the regular UniDip algorithm has terminated. Although these structures are no longer significant enough to be regarded as independent clusters by UniDip, they can still be part of a cluster. Therefore, we mirror the respective area between two clusters and calculate the Dip-p-value. If this indicates that there are still multimodal structures left, we again search for appropriate modes. In order to check whether a found structure matches the adjacent clusters, we apply a strategy that has been described in [6]. Here, the closest $2|S|$ samples of the respective cluster combined with the newly found structure S are used to calculate the Dip-p-value. If this value indicates unimodality, the structure will be added to that cluster and the process is repeated. The described procedure is shown in Algorithm 1. Since in our case a lot of Dip-p-values have to be calculated, a fast calculation of Dip-p-values is favourable.

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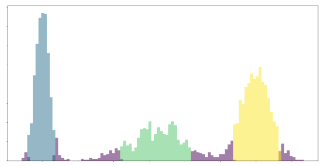
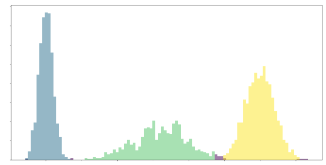
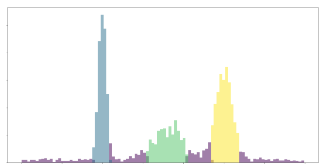


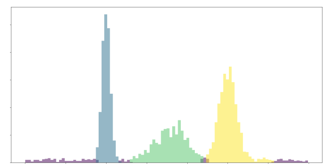
(a) Result of UniDip NMI = 0.81.



(b) Result of TailoredDip NMI = 0.96.



(c) Result of UniDip NMI = 0.70 (noisy data).



(d) Result of TailoredDip NMI = 0.78 (noisy data).

Figure 1: Results of TailoredDip and UniDip on a sample data set consisting of three Gaussian clusters. The identified clusters are coloured in blue, green, and yellow respectively. Outliers are shown in purple.

1.2 Assigning Noise We also present a strategy for assigning outliers to clusters, paying attention to the different tails of the surrounding distributions. In terms of one-dimensional data, it makes sense to define a threshold between every two clusters, indicating whether an outlier is more likely to belong to the left or right cluster. A naive approach would now be to simply set the midpoint between the end of the left and the start of the right cluster. This strategy was chosen in [5], for example. However, this approach completely ignores the existing structures, since it is irrelevant whether a cluster ends abruptly (e.g. in case of an uniform distribution) or fades out slowly (e.g. in case of a normal distribution). To pay attention to these properties, we consider the Empirical Cumulative Distribution Function (ECDF), which is also used to calculate the Dip-value. Here, we draw a straight line from the last point of the left cluster to the first point of the right cluster. In Fig. 2 this is represented by the dotted red line. We now define the intersection of this line with the ECDF of the data as the cluster boundary. Looking at this point in Fig. 2 (left vertical line) we can see that it separates

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Algorithm 1: The TailoredDip algorithm

Input: one-dimensional data set X, significance α
Output: labels

1 // Get initial clusters by running UniDip
2 $labels, k$ = UniDip(X, α)
3 **for** $i = 0;\ i \leq k;\ i \mathrel{+}= 1$ **do**
4   **if** $i == 0$ **then**
5   X_{sub} = samples left of first cluster
6   **else if** $i == k$ **then**
7   X_{sub} = samples right of last cluster
8   **else**
9   X_{sub} = samples between cluster i and $i+1$
10   // Is X_{sub} uniformly distributed (only noise)?
11   X_{mirror} = mirror X_{sub}
12   p =p-value(Dip(X_{mirror}), $|X_{\text{mirror}}|$)
13   **if** $p < \alpha$ **then**
14     $labels_{\text{new}}, k_{\text{new}}$ =UniDip(X_{sub}, α)
15   X_{first} = combine cluster i with the first new cluster // *(ignore if $i == 0$)*
16   p_{first} = p-value(Dip(X_{first}), $|X_{\text{first}}|$)
17   X_{last} = combine cluster $i+1$ with the last new cluster // *(ignore if $i == k$)*
18   p_{last} = p-value(Dip(X_{last}), $|X_{\text{last}}|$)
19   **if** *$i \neq 0$ and $p_{first} \geq \alpha$ and ($k_{new} \neq 1$ or $p_{first} \geq p_{last}$)* **then**
20   Update *labels* by adding all entries with $labels_{\text{new}} == 1$ to cluster i
21   **else if** *$i \neq k$ and $p_{last} \geq \alpha$ and ($k_{new} \neq 1$ or $p_{last} > p_{first}$)* **then**
22   Update *labels* by adding all entries with $labels_{\text{new}} == k_{\text{new}}$ to cluster $i+1$
23   **if** *Cluster i or $i+1$ was updated* **then**
24   **go to** line 4
25 **return** *labels*

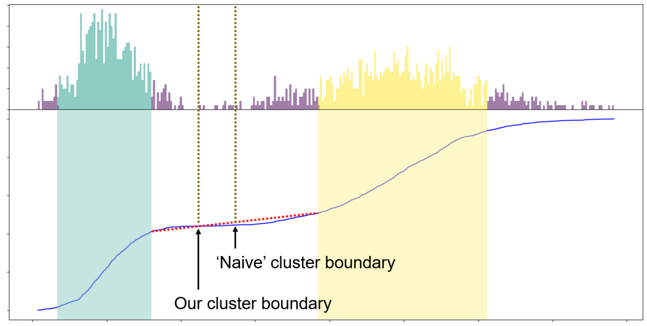

Figure 2: Visualisation of our strategy to assign outliers to the neighbouring clusters. [Top] A histogram of the data. The left (teal) cluster originates from a $\mathcal{N}(0, 1)$ distribution with 700 samples, the right (yellow) cluster originates from a $\mathcal{N}(10, 2.5)$ distribution with 900 samples. The outliers are shown in purple. [Bottom] The ECDF of the data is shown in blue. The areas of the clusters are highlighted in their respective colours. The dotted red line indicates the connection line between the end of the teal and the beginning of the yellow cluster. The right vertical brown line marks the position of a naive cluster boundary, which corresponds to the centre of the red line. Our boundary corresponds to the intersection of the red line with the ECDF and captures the different tails of the two cluster much better.

the tails of the two distributions much better than the naive strategy (right vertical line) since it better captures the higher standard deviation of the right cluster. If more than one intersection occurs, we choose the one closest to the midpoint between the clusters.

2 Additional Information for the Experiments

2.1 Computational Setup Dip'n'Sub as well as the algorithms DipMeans [4], projected DipMeans [1], SkinnyDip [7], DipExt [9], LDA-k-means [2] and SubKmeans [8] are all implemented in Python. Regarding FOSSCLU [3] we use the Java implementation as referenced in the paper. We conduct all runtime experiments on a machine with an Intel Core i7-5600U CPU with 2.60GHz and 8GB RAM. Further, we use Python 3.7 and in case of FOSSCLU, we use Java 8 due to compatibility issues.

2.2 Data Sets We conduct experiments on 9 real world data sets and one synthetic data set (the latter can be seen in the main paper in Fig. 4). Banknotes (BANK), User Knowledge (USER), HTRU2 and Mice Protein (MICE) are numerical data sets from the UCI repository[1]. SonyAIBO (AIBO), MoteStrain (MOTE), Symbols (SYMB) and OliveOil (OLIVE) are time series data sets[2], and ALOI[3] is an image data collection. ALOI was preprocessed as described in [10], resulting in 288 samples devided into 4 clusters. Other than ALOI, no data set did receive any pre-processing, except that features with a variance of 0 were removed. Note, that TailoredDip only works with continuous features, otherwise each value can be recognized as a separate mode. A summary of the data sets is given in Table 1.

[1] https://archive.ics.uci.edu
[2] https://www.timeseriesclassification.com
[3] https://aloi.science.uva.nl/

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Table 1: Summary of the data sets (N = number of data points, d = dimensionality, k = number of clusters).

Dataset	N	d	k
SYNTH	6,300	8	7
BANK	1,372	4	2
USER	403	5	4
HTRU2	17,898	8	2
ALOI	288	66	4
MICE	1,077	68	8
AIBO	621	70	2
MOTE	1,272	84	2
SYMB	1,020	398	6
OLIVE	60	570	4

2.3 Interpolate Look-up Table We would like to briefly explain how missing values in the state-of-the-art look-up table are interpolated. Basically two interpolations must be performed. First, the values for the number of samples that lie below and above the input N must be searched for in the table. By using these two values we are able to interpolate all relevant (Dip,p)-pairs in relation to $\sqrt{N}$. In this interpolated array we search for the Dip-values that are below and above our input Dip to interpolate the Dip-p-value linearly.

2.4 Large Distribution Table In Tables 2 and 3 we show Dip-p-value calculations with the three methods 'table' (T), 'function' (F) and 'bootstrap' (B) for samples of 15 different sample sizes and a total of 23 distribution scenarios. In all cases, we can observe that our fitted function produces basically the same Dip-p-values as the other two methods. For this evaluation we first consider 8 different unimodal distributions:

- $\mathcal{N}(a, b)\dots$ normal distribution with mean a and standard deviation b
- $\mathcal{T}(d, a, b)\dots$ students-t distribution with d degrees of freedom, centre a and scaling b
- $\mathcal{L}(a, b)\dots$ Laplace distribution with centre a and scaling b
- $\mathcal{U}(a, b)\dots$ uniform distribution between a and b
- $\mathcal{G}(s, a, b)\dots$ Gamma distribution with shape parameter s, centre a and scaling b
- $\mathcal{E}(a, b)\dots$ exponential distribution with centre a and scaling b
- $\mathcal{B}(s, r, a, b)\dots$ Beta distribution with shape parameters s and r, centre a and scaling b
- $\mathcal{T}_{nc}(d, c, a, b)\dots$ non central students-t distribution with d degrees of freedom, non centrality c, centre a and scaling b

First, Table 2 shows results for these distributions, with only the listed distributions involved individually. We then generate 8 multimodal distributions by combining $\frac{N}{2}$ samples from one distribution with $\frac{N}{2}$ samples from the same distribution with a different centre. Additionally, we consider 7 cases, where we generate samples by choosing half the points from $\mathcal{N}(4, 1)$ and the other half from one of the other unimodal distributions. These combinations always show multimodal structure. An exception is the case of samples from $\mathcal{N}(4, 1) \cup \mathcal{T}_{nc}(4, 2, 0, 1)$. We include this combination to have a relatively unambiguous case between unimodal and multimodal. Our function performs reliably in all cases as can be seen in Table 3.

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Table 2: Dip-p-values for different unimodal distributions with varying sample sizes N. All given values are averages for 100 random samples ± standard deviation. Respective first, second and third rows per distribution show Dip-p-values calculated with methods ‘table’ (T), ‘function’ (F) and ‘bootstrapping’ (B, 1000 repetitions); ∗: values obtained by $\sqrt{N}-$ interpolation, †: values not available.

Distr.	Meth.	$N=50$	$N=67$	$N=100$	$N=234$	$N=500$	$N=678$	$N=1k$	$N=2345$	$N=5k$	$N=6789$	$N=10k$	$N=23456$	$N=50k$	$N=67890$	$N=100k$
$\mathcal{N}(4,1)$	T	0.77 ± 0.24	0.81 ± 0.21*	0.80 ± 0.20	0.86 ± 0.19*	0.91 ± 0.14	0.94 ± 0.13*	0.94 ± 0.09	0.97 ± 0.07*	0.99 ± 0.03	0.98 ± 0.05*	0.99 ± 0.04	0.99 ± 0.01*	1.00 ± 0.02	1.00 ± 0.01*	†
	F	0.77 ± 0.24	0.81 ± 0.21	0.81 ± 0.20	0.86 ± 0.19	0.91 ± 0.14	0.94 ± 0.13	0.95 ± 0.09	0.97 ± 0.07	0.99 ± 0.03	0.98 ± 0.05	0.99 ± 0.04	1.00 ± 0.01	1.00 ± 0.02	1.00 ± 0.01	1.00 ± 0.02
	B	0.77 ± 0.24	0.81 ± 0.21	0.81 ± 0.20	0.86 ± 0.19	0.91 ± 0.14	0.94 ± 0.13	0.95 ± 0.09	0.97 ± 0.07	0.99 ± 0.03	0.98 ± 0.05	0.99 ± 0.04	1.00 ± 0.01	1.00 ± 0.02	1.00 ± 0.01	1.00 ± 0.02
$\mathcal{T}(4,0,1)$	T	0.78 ± 0.22	0.85 ± 0.19*	0.84 ± 0.20	0.88 ± 0.18*	0.94 ± 0.10	0.95 ± 0.09*	0.97 ± 0.07	0.97 ± 0.07*	0.99 ± 0.04	0.99 ± 0.03*	0.99 ± 0.02	1.00 ± 0.01*	1.00 ± 0.00	1.00 ± 0.01*	†
	F	0.78 ± 0.22	0.85 ± 0.19	0.85 ± 0.20	0.88 ± 0.18	0.94 ± 0.10	0.96 ± 0.09	0.97 ± 0.07	0.97 ± 0.07	0.99 ± 0.04	0.99 ± 0.03	0.99 ± 0.02	1.00 ± 0.01	1.00 ± 0.00	1.00 ± 0.01	1.00 ± 0.00
	B	0.78 ± 0.22	0.85 ± 0.19	0.85 ± 0.20	0.88 ± 0.18	0.94 ± 0.10	0.96 ± 0.09	0.97 ± 0.07	0.97 ± 0.07	0.99 ± 0.04	0.99 ± 0.02	0.99 ± 0.02	1.00 ± 0.01	1.00 ± 0.00	1.00 ± 0.01	1.00 ± 0.00
$\mathcal{L}(0,2)$	T	0.85 ± 0.19	0.88 ± 0.18*	0.92 ± 0.11	0.95 ± 0.11*	0.98 ± 0.04	0.98 ± 0.04*	0.99 ± 0.01	0.99 ± 0.04*	1.00 ± 0.00	1.00 ± 0.00*	1.00 ± 0.00	1.00 ± 0.00*	1.00 ± 0.00	1.00 ± 0.00*	†
	F	0.85 ± 0.19	0.89 ± 0.18	0.92 ± 0.11	0.95 ± 0.11	0.98 ± 0.04	0.99 ± 0.04	0.99 ± 0.01	0.99 ± 0.04	1.00 ± 0.00	1.00 ± 0.00	1.00 ± 0.00	1.00 ± 0.00	1.00 ± 0.00	1.00 ± 0.00	1.00 ± 0.00
	B	0.85 ± 0.19	0.89 ± 0.18	0.92 ± 0.11	0.95 ± 0.11	0.98 ± 0.04	0.99 ± 0.03	1.00 ± 0.01	0.99 ± 0.04	1.00 ± 0.00	1.00 ± 0.00	1.00 ± 0.00	1.00 ± 0.00	1.00 ± 0.00	1.00 ± 0.00	1.00 ± 0.00
$\mathcal{U}(0,2)$	T	0.52 ± 0.29	0.50 ± 0.28*	0.46 ± 0.28	0.53 ± 0.29*	0.52 ± 0.28	0.54 ± 0.31*	0.48 ± 0.30	0.50 ± 0.28*	0.56 ± 0.28	0.54 ± 0.29*	0.52 ± 0.32	0.50 ± 0.30*	0.53 ± 0.28	0.51 ± 0.30*	†
	F	0.52 ± 0.30	0.51 ± 0.28	0.46 ± 0.28	0.53 ± 0.30	0.52 ± 0.29	0.54 ± 0.31	0.48 ± 0.30	0.49 ± 0.29	0.56 ± 0.28	0.54 ± 0.30	0.52 ± 0.32	0.50 ± 0.30	0.53 ± 0.28	0.51 ± 0.30	0.56 ± 0.28
	B	0.52 ± 0.29	0.51 ± 0.28	0.46 ± 0.28	0.53 ± 0.30	0.52 ± 0.29	0.54 ± 0.31	0.48 ± 0.30	0.50 ± 0.29	0.56 ± 0.28	0.53 ± 0.30	0.52 ± 0.32	0.50 ± 0.30	0.53 ± 0.28	0.51 ± 0.31	0.56 ± 0.28
$\mathcal{G}(2,-1,1)$	T	0.76 ± 0.23	0.78 ± 0.22*	0.81 ± 0.21	0.88 ± 0.14*	0.91 ± 0.14	0.91 ± 0.11*	0.93 ± 0.13	0.97 ± 0.08*	0.98 ± 0.05	0.98 ± 0.05*	0.99 ± 0.02	0.99 ± 0.02*	1.00 ± 0.01	1.00 ± 0.00*	†
	F	0.76 ± 0.24	0.79 ± 0.22	0.81 ± 0.21	0.88 ± 0.14	0.91 ± 0.14	0.92 ± 0.11	0.94 ± 0.13	0.97 ± 0.08	0.99 ± 0.05	0.99 ± 0.05	0.99 ± 0.02	1.00 ± 0.02	1.00 ± 0.01	1.00 ± 0.00	1.00 ± 0.00
	B	0.76 ± 0.23	0.79 ± 0.22	0.81 ± 0.21	0.88 ± 0.14	0.91 ± 0.14	0.92 ± 0.11	0.94 ± 0.13	0.98 ± 0.08	0.99 ± 0.05	0.99 ± 0.05	0.99 ± 0.02	1.00 ± 0.01	1.00 ± 0.00	1.00 ± 0.00	1.00 ± 0.00
$\mathcal{E}(0,1)$	T	0.72 ± 0.26	0.79 ± 0.21*	0.79 ± 0.23	0.86 ± 0.18*	0.92 ± 0.11	0.95 ± 0.10*	0.94 ± 0.11	0.98 ± 0.05*	0.99 ± 0.03	0.99 ± 0.02*	1.00 ± 0.01	1.00 ± 0.00*	1.00 ± 0.00	1.00 ± 0.00*	†
	F	0.72 ± 0.26	0.79 ± 0.21	0.80 ± 0.23	0.86 ± 0.18	0.93 ± 0.11	0.95 ± 0.10	0.94 ± 0.11	0.98 ± 0.05	0.99 ± 0.04	1.00 ± 0.02	1.00 ± 0.01	1.00 ± 0.00	1.00 ± 0.00	1.00 ± 0.00	1.00 ± 0.00
	B	0.72 ± 0.26	0.79 ± 0.21	0.79 ± 0.23	0.86 ± 0.18	0.93 ± 0.11	0.95 ± 0.10	0.94 ± 0.11	0.98 ± 0.05	0.99 ± 0.04	1.00 ± 0.02	1.00 ± 0.01	1.00 ± 0.00	1.00 ± 0.00	1.00 ± 0.00	1.00 ± 0.00
$\mathcal{B}(2,2,1,1)$	T	0.65 ± 0.25	0.66 ± 0.25*	0.73 ± 0.25	0.81 ± 0.18*	0.82 ± 0.22	0.84 ± 0.21*	0.84 ± 0.21	0.90 ± 0.16*	0.96 ± 0.07	0.96 ± 0.09*	0.98 ± 0.05	0.98 ± 0.05*	0.99 ± 0.02	0.99 ± 0.01*	†
	F	0.65 ± 0.25	0.67 ± 0.26	0.73 ± 0.26	0.82 ± 0.18	0.82 ± 0.22	0.84 ± 0.21	0.85 ± 0.21	0.90 ± 0.16	0.96 ± 0.07	0.96 ± 0.09	0.98 ± 0.05	0.99 ± 0.05	0.99 ± 0.02	1.00 ± 0.01	0.99 ± 0.02
	B	0.65 ± 0.26	0.66 ± 0.25	0.73 ± 0.25	0.81 ± 0.19	0.82 ± 0.22	0.84 ± 0.21	0.84 ± 0.21	0.90 ± 0.16	0.96 ± 0.07	0.97 ± 0.09	0.98 ± 0.05	0.99 ± 0.05	0.99 ± 0.02	1.00 ± 0.01	1.00 ± 0.02
$\mathcal{T}_{nc}(4,2,0,1)$	T	0.80 ± 0.21	0.78 ± 0.23*	0.85 ± 0.17	0.89 ± 0.14*	0.96 ± 0.06	0.94 ± 0.10*	0.95 ± 0.08	0.98 ± 0.03*	0.99 ± 0.04	0.99 ± 0.02*	0.99 ± 0.01	1.00 ± 0.01*	1.00 ± 0.00	1.00 ± 0.00*	†
	F	0.80 ± 0.21	0.78 ± 0.23	0.85 ± 0.17	0.90 ± 0.14	0.96 ± 0.06	0.95 ± 0.10	0.96 ± 0.08	0.98 ± 0.03	0.99 ± 0.04	0.99 ± 0.02	1.00 ± 0.01	1.00 ± 0.01	1.00 ± 0.00	1.00 ± 0.00	1.00 ± 0.00
	B	0.80 ± 0.21	0.78 ± 0.23	0.85 ± 0.17	0.90 ± 0.14	0.96 ± 0.06	0.95 ± 0.10	0.96 ± 0.08	0.99 ± 0.03	0.99 ± 0.04	0.99 ± 0.02	1.00 ± 0.01	1.00 ± 0.01	1.00 ± 0.00	1.00 ± 0.00	1.00 ± 0.00

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Table 3: Dip-p-values for different combinations of distributions with varying sample sizes N. All given values are averages for 100 random samples $\pm$ standard deviation. Respective first, second and third rows per distribution show Dip-p-values calculated with methods 'table' (T), 'function' (F) and 'bootstrapping' (B, 1000 repetitions). Values are multiplied by 100. $*$: values obtained by $\sqrt{N}-$ interpolation, †: values not available.

Distr.	Meth.	$N = 50$	$N = 67$	$N = 100$	$N = 234$	$N = 500$	$N = 678$	$N = 1k$	$N = 2345$	$N = 5k$	$N = 6789$	$N = 10k$	$N = 23456$	$N = 50k$	$N = 67890$	$N = 100k$
$\mathcal{N}(4,1)$	T	8.94 ± 15.0	6.97 ± 14.1*	4.95 ± 10.4	0.06 ± 0.23*	0.00 ± 0.00	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	†
$\cup$	*F*	8.83 ± 14.9	7.03 ± 14.2	4.87 ± 10.3	0.09 ± 0.25	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00
$\mathcal{N}(0,1)$	B	8.78 ± 15.0	6.81 ± 13.9	4.97 ± 10.5	0.06 ± 0.21	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00
$\mathcal{T}(4,0,1)$	T	21.3 ± 22.7	16.7 ± 23.3*	9.11 ± 16.0	1.25 ± 3.85*	0.01 ± 0.03	0.00 ± 0.01*	0.00 ± 0.00	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	†
$\cup$	F	21.0 ± 22.9	16.8 ± 23.6	9.01 ± 16.0	1.26 ± 3.74	0.01 ± 0.04	0.00 ± 0.01	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00
$\mathcal{T}(4,4,1)$	B	21.0 ± 22.9	16.8 ± 23.7	8.87 ± 15.9	1.21 ± 3.85	0.00 ± 0.03	0.00 ± 0.01	0.00 ± 0.01	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00
$\mathcal{L}(0,2)$	T	24.1 ± 24.9	23.2 ± 25.7*	16.4 ± 23.3	2.36 ± 7.87*	0.02 ± 0.08	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	†
$\cup$	F	23.9 ± 25.1	23.3 ± 26.0	16.2 ± 23.4	2.37 ± 7.82	0.03 ± 0.10	0.00 ± 0.01	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00
$\mathcal{L}(7,2)$	B	23.8 ± 24.8	23.1 ± 25.9	16.4 ± 23.5	2.34 ± 8.06	0.02 ± 0.10	0.00 ± 0.00	0.00 ± 0.01	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00
$\mathcal{U}(0,2)$	T	0.07 ± 0.09	0.01 ± 0.01*	0.00 ± 0.00	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	†
$\cup$	F	0.15 ± 0.13	0.04 ± 0.03	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00
$\mathcal{U}(3,2)$	B	0.08 ± 0.13	0.01 ± 0.03	0.00 ± 0.01	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00
$\mathcal{G}(2,-1,1)$	T	1.00 ± 5.80	0.18 ± 1.28*	0.01 ± 0.04	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	†
$\cup$	F	1.07 ± 5.78	0.21 ± 1.23	0.02 ± 0.06	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00
$\mathcal{G}(2,5,1)$	B	0.98 ± 5.48	0.17 ± 1.11	0.01 ± 0.06	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00
$\mathcal{E}(0,1)$	T	0.08 ± 0.36	0.00 ± 0.02*	0.00 ± 0.00	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	†
$\cup$	F	0.11 ± 0.37	0.01 ± 0.03	0.00 ± 0.01	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00
$\mathcal{E}(4,1)$	B	0.09 ± 0.42	0.00 ± 0.01	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00
$\mathcal{B}(2,2,1,1)$	T	7.05 ± 14.1	4.69 ± 10.4*	1.05 ± 2.26	0.03 ± 0.13*	0.00 ± 0.00	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	†
$\cup$	F	7.03 ± 14.0	4.72 ± 10.4	1.10 ± 2.11	0.05 ± 0.16	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00
$\mathcal{B}(2,2,2,1)$	B	6.80 ± 14.0	4.68 ± 10.4	1.00 ± 2.07	0.04 ± 0.18	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00
$\mathcal{T}_{nc}(4,2,0,1)$	T	0.79 ± 2.20	0.32 ± 2.17*	0.01 ± 0.06	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	†
$\cup$	F	0.83 ± 2.07	0.37 ± 2.16	0.03 ± 0.08	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00
$\mathcal{T}_{nc}(4,2,7,1)$	B	0.74 ± 2.17	0.36 ± 2.42	0.01 ± 0.06	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00
$\mathcal{N}(4,1)$	T	19.0 ± 23.1	9.19 ± 12.4*	7.03 ± 13.7	0.35 ± 0.89*	0.00 ± 0.01	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	†
$\cup$	F	18.8 ± 23.2	9.16 ± 12.4	6.92 ± 13.6	0.38 ± 0.85	0.01 ± 0.02	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00
$\mathcal{T}(4,0,1)$	B	18.7 ± 23.1	9.35 ± 12.6	6.90 ± 13.6	0.33 ± 0.88	0.00 ± 0.01	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00
$\mathcal{N}(4,1)$	T	59.4 ± 32.6	55.5 ± 32.0*	54.0 ± 30.9	44.7 ± 32.9*	25.4 ± 28.0	21.6 ± 23.9*	13.0 ± 17.6	0.89 ± 3.11*	0.01 ± 0.05	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	†
$\cup$	F	59.5 ± 32.9	56.0 ± 32.3	54.0 ± 31.2	44.7 ± 33.2	25.2 ± 28.2	21.4 ± 24.1	12.8 ± 17.6	0.94 ± 3.05	0.02 ± 0.07	0.00 ± 0.01	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00
$\mathcal{L}(0,2)$	B	59.4 ± 32.8	56.0 ± 32.3	54.2 ± 31.1	44.6 ± 33.2	25.4 ± 28.3	21.4 ± 23.9	12.7 ± 17.6	0.85 ± 2.91	0.01 ± 0.03	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00
$\mathcal{N}(4,1)$	T	24.2 ± 26.4	20.0 ± 23.3*	11.2 ± 18.4	1.14 ± 2.95*	0.03 ± 0.18	0.00 ± 0.04*	0.00 ± 0.00	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	†
$\cup$	F	24.0 ± 26.6	20.1 ± 23.5	12.1 ± 18.4	1.17 ± 2.86	0.04 ± 0.19	0.01 ± 0.05	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00
$\mathcal{U}(0,2)$	B	23.8 ± 26.5	20.1 ± 23.5	11.0 ± 18.4	1.08 ± 2.91	0.03 ± 0.16	0.00 ± 0.01	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00
$\mathcal{N}(4,1)$	T	28.8 ± 28.3	28.2 ± 28.3*	28.0 ± 28.5	9.39 ± 16.1*	3.37 ± 8.94	1.07 ± 3.24*	0.13 ± 0.55	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	†
$\cup$	F	28.6 ± 28.5	28.4 ± 28.7	27.9 ± 28.7	9.33 ± 16.1	3.32 ± 8.86	1.05 ± 3.05	0.15 ± 0.54	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00
$\mathcal{G}(2,-1,1)$	B	28.6 ± 28.5	28.5 ± 28.8	28.0 ± 28.6	9.30 ± 16.1	3.31 ± 8.94	1.02 ± 3.00	0.13 ± 0.56	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00
$\mathcal{N}(4,1)$	T	17.6 ± 23.2	9.91 ± 14.5*	9.01 ± 16.6	1.26 ± 4.10*	0.05 ± 0.33	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	†
$\cup$	F	17.5 ± 23.3	9.94 ± 14.6	8.94 ± 16.6	1.29 ± 4.03	0.07 ± 0.31	0.00 ± 0.01	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00
$\mathcal{E}(0,1)$	B	17.5 ± 23.4	9.91 ± 14.7	8.79 ± 16.6	1.26 ± 4.00	0.05 ± 0.30	0.00 ± 0.01	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00
$\mathcal{N}(4,1)$	T	22.3 ± 25.4	19.4 ± 25.5*	10.3 ± 15.5	1.51 ± 3.26*	0.02 ± 0.05	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	0.00 ± 0.00	0.00 ± 0.00*	†
$\cup$	F	22.1 ± 25.5	19.5 ± 25.8	10.1 ± 15.4	1.51 ± 3.09	0.03 ± 0.07	0.00 ± 0.01	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00
$\mathcal{B}(2,2,1,1)$	B	22.1 ± 25.4	19.8 ± 25.8	10.3 ± 15.4	1.47 ± 3.17	0.01 ± 0.05	0.00 ± 0.01	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00	0.00 ± 0.00
$\mathcal{N}(4,1)$	T	69.8 ± 26.9	67.7 ± 27.0*	73.9 ± 22.9	82.8 ± 19.0*	82.9 ± 20.5	88.4 ± 15.8*	90.6 ± 12.5	95.5 ± 7.56*	96.9 ± 8.89	97.2 ± 5.51*	98.5 ± 4.34	99.1 ± 2.97*	99.7 ± 0.44	99.8 ± 0.22*	†
$\cup$	F	69.9 ± 27.2	68.3 ± 27.2	74.1 ± 23.1	83.1 ± 19.0	83.1 ± 20.6	88.7 ± 15.8	90.8 ± 12.5	95.7 ± 7.54	97.0 ± 8.91	97.4 ± 5.52	98.7 ± 4.36	99.3 ± 2.97	99.9 ± 0.43	1.00 ± 0.13	1.00 ± 0.06
$\mathcal{T}_{nc}(4,2,0,1)$	B	69.8 ± 27.1	68.0 ± 27.3	73.7 ± 23.4	83.0 ± 19.2	83.2 ± 20.3	88.8 ± 15.9	90.8 ± 12.6	95.8 ± 7.60	97.0 ± 9.11	97.4 ± 5.59	98.8 ± 4.29	99.4 ± 2.94	99.9 ± 0.36	1.00 ± 0.08	1.00 ± 0.02

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